## **12. Electrostatic field of the circle**

We consider the Laplace equation as the mathematical model of the distribution of the potential of the electrostatic field of a point charge and an infinite wire. Then we solve the Dirichlet problem for the Laplace equation in a circle, using the method of variable separation. We consider the interior and exterior boundary value problem.

### **12.1. Potential of the electrostatic field of a point charge**

The electrostatic field in the space is described by the ***Laplace equation***

 *uxx* + *uyy* + *uzz* = 0, (12.1)

where the function *u = u*(*x*,*y*,*z*)is the potential of electrostatic field at the concrete space point. Let us consider the field of unit point charge. Transform the Laplace equation with using sphere coordinates:

 *x = r* sin θ cos ϕ,  *y = r* sin θ sin ϕ, *z = r* cos θ. (12.2)



Find the partial derivatives



Then we can find the second derivatives and put the result to the formula (12.1). After transformations, we determine the Laplace equation in the sphere coordinates

 . (12.3)

The equation (12.3) is more difficult than the initial Laplace equation (12.1). However, return to the given problem statement. We analyze the electrostatic field of the point charge. In this situation, the potential of the field at the control point depends only of the distance between this point and the charge, because of the isotropy of the space. Therefore, the derivatives of the potential with respect to the angle coordinates are equal to zero. Then the equation (12.3) can be transformed to the following form.

 . (12.4)

This equation the following the following general solution



Consider the easiest case *c*1 = 1, *c*2 = 0. Then we have the formula

  (12.5)

This is the solution of the Laplace equation with spherical symmetry. This is called the ***fundamental solution of the Laplace equation in the space***. This is the potential of the electrostatic field of the unit point charge. If we have the charge *e*, then the solution is



By this formula the potential of the field tends to infinity as *r* tends to zero; it tends to zeroas *r* tends to infinity.



### **12.2. Potential of the electrostatic field of an infinite wire**

Consider now the electrostatic field of the infinite wire. Determine the cylinder coordinates

 *x = r* cos ϕ,  *y = r* sin ϕ, *z = z*. (12.6)



After changing the variable, we obtain the Laplace equation in the cylinder coordinates

  (12.7)

We analyze the field of the infinite wire. Then the potential of the field at the concrete point depends only from the distance between this point and the wire, i.e. from the coordinate *r.* Therefore, its derivatives with respect to the angle and the coordinate *z* is equal to zero. Now the equality (12.7) is transformed to

 . (12.8)

The general solution of this equation is



Consider the partial case *c*1 = -1, *c*2 = 0. Then we have the formula

  (12.9)

This function is called the ***fundamental solution of the Laplace equation in the plane***. If we have the wire with charge density *e*, then the potential of this field is



By this formula, the potential of the field tends to infinity as *r* tends to zero; it tends to zeroas *r* tends to infinity.



### **12.3. Dirichlet problem for the Laplace equation in circle**

Let us have the Laplace equation

 *uxx* + *uyy* = 0 (12.10)

 in the circle of the radium *a.* Use the polar coordinates

 *x = r* cos ϕ,  *y = r* sin ϕ. (12.11)

The Laplace equation in the polar coordinates is

  (12.12)

The value of the function *u* at the boundary of the circle is given

 *u*(*a*,*ϕ*) = *f*(*ϕ*), (12.13)

where *f* is known function. This first problem is called ***Dirichlet problem***. In reality, we consider

the interior and exterior Dirichlet problem. For the interior problem the Laplace is considered in the circle, i.e. for the values 0 < *r* < *a.* For the exterior problem the Laplace is considered outside of the circle, i.e. for the values *a* < *r* < ∞.

### **12.4. Method of separation of variables**

We solve both problems by the method of the separation of variables. Determine the solution of the problem (12.12), (12.13) by the formula

 *u*(*r*,*ϕ*) = *R*(*r*)Ф(*ϕ*). (12.14)

Put the function *u* from the formula (12.14) to the equality (12.12). We have



Divide this equality by *R*Ф/*r*. We get



The equality of the functions of different variables can be true, if these functions are equal to a constant. Denoting this constant by *λ*, we obtain to ordinary differential equations

  (12.15)

  (12.16)

It is necessary to add a boundary condition. Note that the solution of our problem is periodic with respect to the angle, i.e. the following equality holds

 *u*(*r*,0) = *u*(*r*,2*π*)*,* (12.17)

After putting the value of the function *u* from the formula (12.14) to the equality (2.17), we get

 Ф(0) = Ф(2*π*). (12.18)

This is the boundary conditions for the equation (12.16).

### **12.5. Sturm–Liouville problem**

The boundary problem (12.16), (12.18) has the trivial solution. In this case, the function *u* is zero by the formula (12.14). However, this contradicts the boundary condition (12.13). Therefore, it is necessary to choose the constant *λ* such the solution of the problem (12.16), (12.18) is non-zero. This is the Sturm–Liouville problem.

Consider the equation (12.16). If *λ* is negative, then the general solution of this equation is



where *c*1 and *c*2 are constants. Put this result the formula (12.18). We have



This result can be true, only if *c*1 = *c*2 = 0. In this case Ф=0. But in this case *u* = 0 that is impossible.

Suppose *λ =* 0. Then the general solution of the problem (12.16), (12.18) is



Check the condition (12.18). We have



Therefore, *c*2 = 0, i.e. Ф is constant. Thus, the value *λ =* 0 is possible.

Let now *λ* be positive. Then the general solution of the problem (12.16), (12.18) is



Put this value to the equality (12.18). We obtain



This is true, if  where *n* is an arbitrary natural number. Thus, we have the infinite set of non-zero solution of the problem (12.16), (12.18)

  (12.19)

where *c*1*n* and *c*2*n* are arbitrary constants.

### **12.6. Solution of the Laplace equation**

Return to the consideration of the equation (12.15) for the value *λ = n*2, *n =* 0,1,… . We have

  (12.20)

Find the solution of this equation by the formula

  (12.21)

where *an* and *bn* are arbitrary constants. Indeed, we calculate









Thus, the solution of the equation (12.20) is determined, in reality, by the formula (12.21).

We have the question, have we can find to groups of constants by the unique given boundary (12.13)? However, we can an additional boundary condition. Suppose we consider the interior boundary problem, i.e. 0 < *r* < *a.* Pass to the limit in the equality (12.21) as *r* tends to zero. Then the value *Rn* tend to infinity, and the function *u* too, because of the equality (12.14), if *bn* is non-zero. This situation is impossible. Therefore, we determine *bn* = 0. Thus, the solution of the equation (12.20) for the interior problem is

  (12.22)

Now we consider the exterior problem, i.e. *a*< *r* < ∞. Pass to the limit in the equality (12.21) as *r* tends to infinity. Then the value *Rn* tend to infinity, and the function *u* too, because of the equality (12.14), if *an* is non-zero. This situation is impossible. Therefore, we determine *an* = 0. Thus, the solution of the equation (12.20) for the interior problem is

  (12.23)

Using the equalities (12.19) and (12.22), find the solution of the Laplace equation for the interior problem by the problem

  (12.24)

where *αn* and *βn* are arbitrary constants. Analogical formula for the exterior problem is

  (12.25)

Determine its sums

  (12.26)

  (12.27)

The formula (12.26) gives the periodic solution of the Laplace equation that is bounded at zero. The formula (12.27) gives the periodic solution of the Laplace equation that is bounded at infinity.

### **12.7. Solution of the Dirichlet problems**

Put the function *u* from the formula (12.26) to the boundary condition (12.13). We get



Multiply this equality by sin*kϕ* and integrate by *ϕ*. We have

  (12.28)

Determine the integral



If *k* is not equal to *n*, we determine



For *k = n* we have



Now we calculate



 If *k* is not equal to *n*, we determine



For *k = n* we have



Put the results to the formula (12.28). We find

  (12.29)

Now we multiply the equality (12.26) by cos*kϕ* and integrate by *ϕ*, *k =* 1,2,… . We have

  (12.30)

We now that



If *k* is not equal to *n*, we determine



For *k = n* we have



Then we get

  (12.31)

Finally, integrate the equality (12.26). We determine



Then we find

  (12.32)

Thus, the solution of the interior Dirichlet problem for the Laplace equation is determine by the formula (12.26), where the Fourier coefficients are determine by the formulas (12.29), (12.31), (12.32).

Using the analogical method, we find the solution of the exterior Dirichlet problem for the Laplace equation is determine by the formula (12.27) with Fourier coefficients

  (12.33)

  (12.34)

  (12.35)

### **12.8. Example. Field potential of the charge circle**

Consider the interior Dirichlet problem with parameters *a =* 1, *f*(*ϕ*) = sin*ϕ*. In this situation we find *α*1 = 1, *αk*= 0 for *k*>1, *β k*= 0 for all *k.* Then we have the solution of the problem



Determine

,



Then the considered function is the solution of the Laplace equation. Besides,  Thus, we found, in reality the solution of the given interior Dirichlet problem.

What is a physical sense of this result? We consider the electrostatic field of the charged circle by the law *f*(*ϕ*) = sin*ϕ*. The potential of the field has the dependence sin*ϕ* with respect to the angle. However, the coefficient before sinus decrease with increasing of the length of the point from the boundary. This potential at the point tends to zero as the point tends to the center of the circle.



Figure 12.1. Potential distribution.

The solution of the problem for the arbitrary radium *a* is



Thus, the potential at the concrete point is in inverse proportion from the radium of the circle.

### **Conclusions**

* The potential of the electrostatic field is described by the Laplace equation.
* The electrostatic field of the point charge is analyzed by the Laplace equation by the sphere coordinates.
* The fundamental solution of the Laplace equation in the space is the potential of the electrostatic field of the point charge.
* The electrostatic field of the infinite wire is analyzed by the Laplace equation by the cylinder coordinates.
* The fundamental solution of the Laplace equation in the plane is the potential of the electrostatic field of the infinite wire.
* The Laplace equation in the circle is analyzed, using polar coordinates.
* There exists interior and exterior Dirichlet boundary problem for the Laplace equation.
* The Laplace equation in the circle is solved by the method of the separation of variables.
* The Sturm – Liouville problem with periodic boundary condition is obtained after using the method of the separation of variables.
* The solutions of interior and exterior Dirichlet problem for the Laplace equation are determined as the Fourier series.
* The Fourier coefficients of the solution are determined, using the given boundary condition.
* The electrostatic field of the charged circle is considered as an example.

### **Task. Field potential of the charge circle**

Consider the electrostatic field of the charged circle of the radium *a*. This phenomenon is described by the Laplace equation in the polar coordinates



with boundary condition

*u*(*a*,*ϕ*) = *f*(*ϕ*).

The boundary problem can be interior or exterior.

Table of parameters

|  |  |  |  |
| --- | --- | --- | --- |
| **variant** | **problem** | ***a*** | ***f* (*ϕ*)** |
| 1 | interior | 2 | - sin *x* |
| 2 | exterior | 3 | cos *x* |
| 3 | interior | ½ | sin 2*x* |
| 4 | exterior | 2 | -sin *x* |
| 5 | interior | 3 | -cos *x* |
| 6 | exterior | ½ | -sin 2*x* |
| 7 | interior | 2 | cos 2*x* |
| 8 | exterior | ½ | -cos 2*x* |

Task:

1. Find the solution of the problem.
2. Check that this is, in reality the solution.
3. Show the graph (potential at the different points of circle).
4. Give the physical interpretation of the results.